Credibility, Efficiency, and Stability: A Theory of Dynamic Matching Markets [*]

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Abstract

We introduce a new dynamic framework to analyze two-sided matching interactions that occur repeatedly over time, such as teacher-student matching or hospital-intern markets in Britain. We propose a new dynamic concept of credible group stability and show that implementing a men-optimal stable matching in each period is credibly group-stable. The result holds for a women-optimal stable matching. A credibly group-stable dynamic matching is individually rational and immune to any defensible group deviations with an appropriate definition of defensibility. We obtain several policy implications for market design. Moreover, a sufficient condition for Pareto efficiency is given for finitely repeated markets.

Keywords: Dynamic matching market; Credibility; Efficiency; Group stability
JEL classification: C71, C78

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1 Introduction

Much of economic life involves two-sided matching that often spans a long horizon. Examples include most teacher-student interactions such as music lessons, business relationships between firms, and hospital-intern markets.

For example, consider music lessons organized by an institution such as City Music Center of Duquesne University in Pittsburgh, PA.[1] The Center's teachers have preferences over students they would like to teach, and students have preferences over teachers. Moreover, to better play a musical instrument, students have to spend many years taking lessons, and thus they need to be involved in long-term relationships. Hence, this is a dynamic two-sided matching market.

For another example, consider British entry-level medical labor markets. These markets involve graduating medical students and teaching hospitals. Students seek residency positions for both medical and surgical programs of hospitals—they have one for the first six months and another for the next six months. Teaching hospitals fill positions in both of these periods. In each period, the market is a many-to-one matching interaction, since students accept at most one hospital and hospitals accept many students. Moreover, since this interaction repeats twice, it is a dynamic many-to-one matching market with two periods. However, it has been modeled as a "static" matching market (See Roth (1991)).

Until now, although static relationships have been extensively studied in matching markets (cf. Roth and Sotomayor (1990) and Roth (2002)), there has been almost no attempt to analyze dynamic relationships.[2] We introduce a new framework to analyze two-sided dynamic interactions: Time is discrete with either finite or infinite horizon. There are two finite disjoint sets of agents. Each agent is supposed either to be matched with those in the opposite set or to be unmatched in each period. There are no frictions: agents do not have to commit themselves to their prior partners and can freely change partners at any period. Each agent has a time-separable utility function over those in the opposite set and being unmatched in each period. The preferences may *vary* across periods.

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[1] See http://www.cmcpgh.org. Tuition does not play a decisive role in matching, because the tuition is not differentiated by teachers or students.

[2] See recent exceptions: Damiano and Lam (2005) and Kurino (2009a) for two-sided matching markets, and Abdulkadiroğlu and Loertscher (2007), Bloch and Cantala (2008), Kurino (2009b) and Ünver (2007) for house allocation problems.

In a related paper, Damiano and Lam (2005) consider the finite horizon model where the preferences are constant across time with a discount factor; that is, finitely repeated matching markets. While this is a useful benchmark, it can be unrealistic for some applications. For example, in the example of the music lessons discussed above, as students' skills improve, they prefer teachers with different skills. Violin teachers may not value students who did not learn the piano in the past. That is, their current preferences may depend on the past matchings. Moreover, Damiano and Lam (2005) assume that agents choose an outcome path, or a sequence of matchings but not a contingent plan based on realized matchings. This is restrictive, because agents can change prior partners at any time. In this paper, we consider a contingent plan called a "dynamic matching." The problem in dynamic matching markets is to analyze what kinds of matchings might arise in each period under a dynamic matching.

In static settings, it has been shown that a property known as "stability" is central to determining whether static matchings will be sustainable in real-life applications (cf. Roth (1984, 1991, 2002)). Stability (Gale and Shapley, 1962) requires that (1) no individual would rather stay unmatched than continue with her current partner, and (2) no pair of individuals such as a teacher and a student or a hospital and an intern, would prefer each other to their current partners. Two stable matchings have attracted much attention in real-life applications as well as in theoretical work: "hospital-optimal" and "intern-optimal stable" matchings in the case of hospital-intern markets, where the former (the latter) is the best stable matching for hospitals (interns) which is at the same time the worst stable matching for interns (hospitals). For example, several regional markets in the aforementioned British markets use "hospital or intern optimal (statically) stable" mechanisms in their centralized matching process, although the markets are dynamic. As Roth (p430, 1991) noted, this static stable mechanism may produce a "higher-order" instability regarding dynamic aspects. In fact, as we will show in Examples 2 and 3, such matchings need not create "dynamically stable" or even "Pareto efficient" outcomes. However, these centralized clearinghouses have been successfully used for the last forty years in Britain. This creates a puzzle: Why is implementing a hospital-optimal or intern-optimal (statically) stable matching so robust in the British markets? This paper provides a theoretical explanation for the robustness.

In this paper, we are concerned with one-to-one matching markets, conventionally called marriage markets (Gale and Shapley, 1962). In a marriage market there are, so called, "men" and "women," each of whom can be matched with at most one partner of the opposite sex. Although we do not deal with many-to-one matching markets such as hospital-intern and teacher-student markets, conceptual tools and insights developed in this paper can be applied to such markets. The aforementioned British markets have been modeled as many-to-two matching markets (Roth, 1991). The hospital or intern-optimal stable matchings correspond to "men or women-optimal stable" matchings in marriage markets.

Traditionally, the cooperative solution concept known as the "core" has been used in analyzing such markets. We begin by pointing out that coalitional deviations considered in the definition of the core are restrictive in dynamic matching markets. Taking into account more general deviations, we propose a definition of (dynamic) group stability that is stronger than the core. An outcome path, or a sequence of matchings, is in the core if no deviating coalition, by choosing another outcome path only among themselves, can make each agent strictly better off. In other words, after the deviation in the first period, agents in a deviating coalition must be matched with each other from the beginning to the end, and are not allowed to be matched with agents outside the coalition. This notion of a deviation is restrictive.

We propose another concept that allows for more general deviations than those permitted in the core, since in the dynamic relationships we explore, we assume that agents are free to sequentially form new partnerships whenever they want. We define "(dynamic) group stability" by requiring a dynamic matching to be immune against group deviations that do not force agents to be matched within the group during all periods.[3] However, a group stable dynamic matching may not always exist (cf. Examples 2 and 3). This means that a dynamic matching consisting only of men-optimal stable matchings may not be group stable in a dynamic setting. We then introduce a new dynamic stability concept called "credible group stability," and show that such a dynamic matching is justified. That is, we show in Proposition 4 that the dynamic matching that assigns a men-optimal stable matching in each period is "credibly group-stable." Similarly, the result holds for women-optimal stable matchings. The hospital-optimal (or intern-optimal) stable mechanism in the aforementioned British markets turns out to be credibly group-stable if we translate it to marriage markets.

Closely looking at possible group deviations from a dynamic matching, we notice that some of them may not be *defensible* in a certain way. Even if a group benefits by reorganizing its match, some members may have an incentive to deviate further by matching with the other agents inside or "outside" the group. In this case, we say that such group deviations are not

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[3] The word "group" is used as a synonym of coalition that is a collection of agents. The use depends on which solution concept is used. Coalition is used for the characteristic function approach such as the core, while group is for the non-characteristic function approach such as group stability.

"defensible." A "credibly-group stable" dynamic matching is immune against any defensible group deviations, and individually rational (i.e. no agent would rather stay unmatched than her current mate).

Our results on credible group stability have significant policy implications. Since a men-optimal (women-optimal) stable matching is favorable to men (women) but not to women (men), we can think of two compromises: 1) choose men-optimal and women-optimal stable matchings alternately, 2) choose a median stable matching in each period that is neither men-optimal nor women-optimal stable. However, both of compromises may not be credibly group-stable (cf. Example 5). Moreover, static many-to-many markets can be alternative to dynamic markets under restricted preference domains. Konishi and Ünver (2006) show that in a many-to-many market, the set of pairwise stable matchings is equivalent to the set of "credibly group-stable" matchings (their notion of credibility is different from ours) under reasonable preference domains. That is, a stable matching other than hospital-optimal (or student-optimal) ones is supported by their credible group-stability but may not be supported by our notion of credibility (cf. Example 5).

The second question we explore is on Pareto efficiency. This question does not arise in a static stable mechanism, since a stable matching is always Pareto efficient in a static market. However, this is not true even for finitely repeated markets (cf. Example 2). Hence, we look at finitely repeated markets and examine whether a credibly group-stable dynamic matching that involves a men-optimal (or women-optimal) stable matching in each period is Pareto efficient. We then introduce a condition, called the "regularity condition," and show in Theorem 3 and Corollary 1 that under this condition, such dynamic matchings are also Pareto efficient.

1.1 Related literature

In a closely related paper, Damiano and Lam (2005) consider finitely repeated marriage models. They propose variants of "core"-like solution concepts by taking into account dynamic commitment and credible deviations. Their model studies exclusively "repeated" markets in which preferences are "time independent." Our model explores "dynamic" relationships that may have changes of preferences as in several real-life markets. That is, the dynamic markets are "time dependent." In this sense, our model incorporates theirs. In the framework of random matching models of money (Kiyotaki and Wright, 1989), Corbae, Temzelides and

Wright (2003) consider endogenous matching by using a solution concept that is immune to one-shot pairwise deviations. The companion paper (Kurino, 2009a) examines this solution concept in our framework. In addition, Abdulkadiroğlu and Loertscher (2007), Bloch and Cantala (2008), Kurino (2009b) and Ünver (2007) study another dynamic matching model of house allocation. Roth and Vande Vate (1990) study a static market to see how, starting from an arbitrary matching, decentralized dynamic process reaches stable matchings.

British medical markets have been modeled as a static many-to-two matching market in that medical students look for two positions and hospitals fill many positions (Roth, 1991). This suggests that static many-to-many markets can be used for a dynamic market. However, this modeling involves strong preference restrictions. For many-to-many matching markets, see Sotomayor (1999), Echenique and Oviedo (2006) and Konishi and Ünver (2006).

In a static setting, the matching literature uses group stability instead of the core as a solution concept because the deviation considered in the definition of the core is not realistic. In other words, the non-characteristic function approach is used to define group stability. For example, see Roth and Sotomayor (1990) for many-to-one matching markets and the papers listed in the previous paragraph for many-to-many matching markets. This approach is also used in network games (Jackson and Wolinsky, 1996).

The credibility problem for deviating coalitions has been studied in both static and dynamic settings. In a static setting, the various bargaining sets have been proposed for games in coalitional form since Aumann and Maschler (1964). The idea is to consider an *objection* to an outcome by a coalition, and a *justified* objection in which some member of the coalition can not form a *counterobjection* consisting of members insider or outside the coalition. An outcome in the *bargaining set* has no justified objections. Zhou (1994) introduces the new *bargaining set*. Klijn and Massó (2003) apply Zhou's definition to the marriage model. Moreover, they introduce *weak stability* and investigate the relation with the bargaining set. These two concepts allow members of a deviating coalition to deviate further by matching with agents inside or outside the coalition. We follow the same approach. In fact, weak stability coincides with credible pairwise stability in a static setting that is a special case of our credible group stability in dynamic settings. On the other hand, Konishi and Ünver (2006) require a deviating coalition to have no further pairwise deviation within the coalition in their definition of credible group stability in many-to-many matching problems. Turning to other approaches in a static setting, Bernheim et al. (1987) propose the concept of *coalition-proof Nash equilibrium* for normal form games. Ray (1989) defines the cooperative analogue of this approach called *modified core*. These concepts require a deviating coalition to have no further deviations within the coalitions, where the further deviations satisfy the same requirement. In the same spirit, Bernheim et al. (1987) define *perfect coalition-proofness* for extensive form games. Damiano and Lam (2005) define the cooperate analogue of *self-sustaining stability* for finitely repeated matching markets.

2 The Model

2.1 Preliminaries: static marriage markets

We define a **static (marriage) market** as a triple $(M, W, \{u_i\}_{i \in I})$. By a static market, we always mean a static marriage market. The set $I := M \cup W$ of agents is divided into two finite disjoint subsets M and W. M is the set of men and W is the set of women. Note that $|M| \neq |W|$ in general. Generic agents are denoted by $i \in I$, while generic men and women are denoted by m and w, respectively. Man m's utility function is $u_m : W \cup \{m\} \to \mathbb{R}$, and woman w's utility function is $u_w : M \cup \{w\} \to \mathbb{R}$. Woman w is **acceptable** to man m if $u_m(w) \geq u_m(m)$, and similarly for m. An agent is said to have **strict preferences** if he or she is not indifferent between any two choices. *We assume throughout the paper that all agents have strict preferences.* In this market, each agent is either matched with another agent of the opposite sex or is unmatched. An outcome is a **matching** defined by a bijection $\mu : M \cup W \to M \cup W$ such that for each $i \in I, (\mu \circ \mu)(i) = i$, and if $\mu(m) \neq m$ then $\mu(m) \in W$, and if $\mu(w) \neq w$ then $\mu(w) \in M$. Fixing M and W, let $\mathcal{M}$ be the set of all matchings. If $\mu(i) = i$, agent i is said to be **unmatched**, and denote this pair by (i, i). If $\mu(m) = w$, equivalently $\mu(w) = m$, then w is said to be **matched** with m, and denote this pair by (m, w). For notational simplicity, we often use $u_i(\mu)$ instead of $u_i(\mu(i))$. A matching μ is **individually rational** if each agent is acceptable to his or her partner, i.e., $u_i(\mu) \geq u_i(i)$ for each agent i in I. Given a matching μ, a pair (m, w) **blocks** μ if they are not matched with each other in μ but prefer each other to their matched partners in μ, i.e. $u_m(w) > u_m(\mu)$ and $u_w(m) > u_w(\mu)$.

Definition 1 (Gale and Shapley (1962)). A matching μ is called **(statically) stable** if it is individually rational, and is not blocked by any pair (m, w) in $M \times W$.

The adverb "statically" is omitted if there is no confusion. Moreover, Gale and Shapley

(1962) prove the existence of stable matchings:

Theorem 1 (**Existence: Gale and Shapley (1962)**). *A stable matching exists for each static market. In particular, when all agents have strict preferences, there always exist a men-optimal stable matching (that every man likes at least as well as any other stable matching) and a women-optimal stable matching.*

2.2 Dynamic marriage markets

We consider a **dynamic (marriage) market** in which one-to-one matching interactions occur repeatedly over time. By a dynamic market, we always mean a dynamic marriage market. Time is discrete with either finite or infinite horizon. We denote the horizon by T. $T < \infty$ stands for a finite horizon, while $T = \infty$ stands for infinite horizon. In this market, there are fixed sets of M and W, where M and W are disjoint and finite. In general, $|M| \neq |W|$. Each agent is supposed either to be matched with at most one agent of the opposite sex or to be unmatched at each period $t = 0, \cdots, T$. There are no frictions: agents do not have to commit themselves to their prior partners and can freely change partners at any period. Each agent has a time-separable utility function over those of the opposite sex and being unmatched. Man m's utility function at period t is given by $u_m^t : W \cup \{m\} \to \mathbb{R}$, while woman w's utility function is $u_w^t : M \cup \{w\} \to \mathbb{R}$. *We assume throughout the paper that all agents have strict preferences in each period.* An **outcome path** is a sequence of matchings in $\mathcal{M}$, denoted by $\boldsymbol{\mu} := \{\mu^t\}_{t=0}^T$. Given an outcome path $\boldsymbol{\mu} = \{\mu^t\}_{t=0}^T$, agent i's utility function is given by

$$U_i(\boldsymbol{\mu}) \;:=\; \sum_{t=0}^{T} u_i^t(\mu^t),$$

where for notational simplicity we use $u_i^t(\mu^t)$ instead of $u_i^t(\mu^t(i))$. We assume that for an infinite horizon case, $U_i(\boldsymbol{\mu})$ is well-defined for any outcome path $\boldsymbol{\mu}$. Each agent knows his or her utility functions as well as those of the other agents. The above structure is common knowledge. Thus, a dynamic market is a triple $(M, W, \{u_i^t\}_{i \in I, t=0,\cdots,T})$. Looking at period t, $(M, W, \{u_i^t\}_{i \in I})$ is a static market, called a **period t (marriage) market**. If we do not need to specify the period, we call it a **constituent (marriage) market**. A dynamic market is called a **repeated (marriage) market** if for each agent $i \in I$ there is a discount factor $\delta_i \in (0, 1]$ and a utility function u_i such that $u_i^t = \delta_i^t u_i$ for each period $t = 0, \cdots, T$.

3 Dynamic Group Stability

3.1 Core and dynamic group stability

In this dynamic market, the problem is which matchings might arise in each period. In other words, which outcome paths[4] will result from interaction among agents? The core[5] gives an answer:

Definition 2. *1.* An outcome path $\boldsymbol{\mu} = \{\mu^t\}_{t=0}^T$ is in the **core** if no coalition blocks it, i.e. there is no coalition A and outcome path $\hat{\boldsymbol{\mu}} = \{\hat{\mu}^t\}_{t=0}^T$ such that
(a) $\hat{\mu}^t(i) \in A$, for each $t = 0, 1, \cdots, T$ and for each i in A, and
(b) $U_i(\hat{\boldsymbol{\mu}}) > U_i(\boldsymbol{\mu})$, for each i in A.

2. It is **individually rational** if for each i in I, $U_i(\boldsymbol{\mu}) \geq \sum_{t=0}^T u_i(i)$.

We will point out that the core is unrealistic by examining deviations that the core concept considers, and then consider a newly defined deviation to define group stability.[6]

Let's examine the core more closely. Condition (a) in Definition 2 requires that after a coalition deviates from $\boldsymbol{\mu}$, all agents in the coalition must be matched only among themselves "from the beginning to the end." On the other hand, condition (b) says that each agent in A is strictly better off in $\hat{\boldsymbol{\mu}}$ than in $\boldsymbol{\mu}$.

Condition (a) is clearly restrictive. We can think of situations in which agents are matched among themselves for only "several" periods, while still being matched with the old partners at other dates. The following example illustrates this point.

Example 1. Consider a two-period dynamic market with $M = \{m_1, m_2\}$ and $W = \{w_1\}$. The constituent markets are illustrated in Figure 1, while the total utilities depending on the outcome paths are shown in Figure 2. In Figure 1, the nodes represent the agents, the lines (or no line) represent matches (or no match). The number attached to a node stands for the utility from the match. In this market, there are two outcome paths in the core: $\boldsymbol{\mu}_1 := (\mu_a, \mu_b)$

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[4] Damiano and Lam (2005) call an outcome path a matching plan.

[5] In general, the core may be empty in our model. An example is given in the Appendix.

[6] This kind of approach has been taken in the matching literature, as we discussed in the Related Literature section.

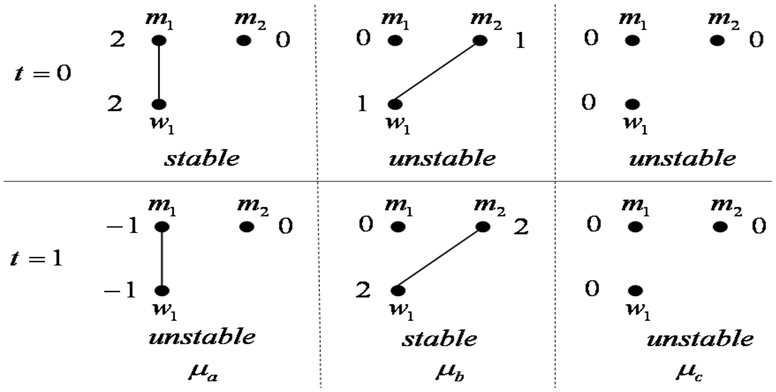

Figure 1: Constituent marriage markets in Example 1

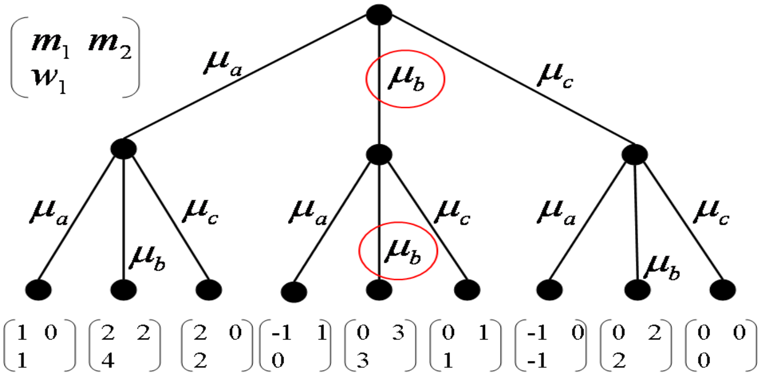

Figure 2: Total utilities in Example 1

and $\boldsymbol{\mu}_2 := (\mu_b, \mu_b)$ the latter of which is indicated by circles in Figure 2. Consider why $\boldsymbol{\mu}_2$ is in the core. We can see that it is individually rational and that no grand coalition blocks it. Consider the coalition $\{m_1, w_1\}$. The outcome paths this coalition can achieve are (μ_a, μ_a), (μ_a, μ_c), and (μ_c, μ_a). Agents (m_1, w_1) obtain $(1, 1)$, $(2, 2)$ and $(-1, -1)$ instead of $(0, 3)$, respectively. However, given that μ_b is chosen at period 1, the pair (m_1, w_1) has an incentive to be matched (i.e. the resulting matching is μ_a) in period 0 and μ_b in period 2. Then, (m_1, w_1) gets $(2, 4)$ instead of $(0, 3)$. Our point is that, instead of requiring that a coalition should be matched only among themselves from the beginning to the end, it may be more appropriate to think that deviators are matched among themselves in only several periods, while still being matched with the old partners in other periods if this results in a superior outcome. We consider these kinds of deviations in the definition of a new solution concept of dynamic group stability. $\square$

Once we allow this kind of deviation, agents become concerned with a contingent plan based on histories of matchings instead of an outcome path. The contingent plan is called a *dynamic matching.*[7] Now we are away from a characteristic function approach,[8] so we use a "group" instead of a coalition for the name of a collection of agents. Our goal is to define dynamic group stability which is "stable" against "group deviations" described above. We need to introduce some new notions:

A **history** at period t, $t \geq 1$, is $h^t := (\mu^0, \mu^1, \cdots, \mu^{t-1}) \in \mathcal{M}^t$, and $h^0 := \emptyset$ is the history at the start of the market. Let $\mathcal{H}^t$ be the set of all histories at period t, i.e. $\mathcal{H}^t = \mathcal{M}^t$. The set of all histories is $\mathcal{H} := \cup_{t=0}^{T} \mathcal{H}^t$.

Definition 3. A **dynamic matching** is a function $\phi : \mathcal{H} \to \mathcal{M}$. Moreover, it is called **history-independent** if in each period, a matching specified by the dynamic matching is independent of histories, i.e., for each $t = 0, 1, \cdots, T$ and for each $h^t, \tilde{h}^t$ in $\mathcal{H}^t$, $\phi(h^t) = \phi(\tilde{h}^t)$.

Note that history independence means that matching in each period is a function of the calender time alone, and that matchings need not be constant across periods. A dynamic matching ϕ induces a unique outcome path $\boldsymbol{\mu}(\phi) := \{\mu^t(\phi)\}_{t=0}^{T}$ recursively as follows: $\mu^0(\phi) := \phi(\emptyset)$, for $t \geq 1$, $\mu^t(\phi) := \phi(\mu^0(\phi), \cdots, \mu^{t-1}(\phi))$. Given ϕ, each agent i's utility function is obtained as $U_i(\phi) := U_i(\boldsymbol{\mu}(\phi))$.

We are interested in whether a given dynamic matching is "stable" (in some sense) against group deviations. To this end, when some group deviates at some history from a given dynamic matching, we must specify how the outsiders respond to the group deviation. This is because the payoffs that agents within the deviating group obtain depend on the outsiders' behavior through the change in histories. In this regard, we make a simple assumption that the outsiders who were matched with agents in the group before the deviation become unmatched, and the other outsiders are matched with the same partners as before. With this in mind, we begin to describe how a matching changes in response to a group deviation.

Definition 4. [9] Given a matching μ, a **(static) group deviation** from μ is a pair $(A, \hat{\mu})$ consisting of a group A and a matching $\hat{\mu}$ such that

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[7] Corbae, Temzelides and Wright (2003) also consider this kind of contingent plan.

[8] Non-characteristic function approaches have been widely used in the many-to-one, many-to-many matching problems and network games, as we discussed in the Related Literature section.

[9] A special case of pair deviations (the group consisting of one man and one woman) coincides with the

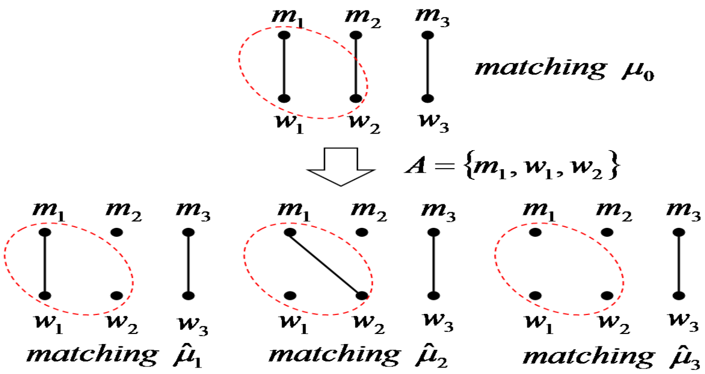

Figure 3: Possible static group deviations by $\{m_1, w_1, w_2\}$

(a) for each i in A, $\hat{\mu}(i) \in A$,
(b) for each i, j in $I \setminus A$, if $\mu(i) = j$, then $\hat{\mu}(i) = j$, and
(c) for each i in A and for each j in $I \setminus A$, if $\mu(i) = j$, then $\hat{\mu}(j) = j$.

The adjective "static" is omitted when there is no confusion. Condition (a) requires that deviating agents in A should be matched with each other. Condition (b) requires that agents outside the group A should be matched according to μ, while condition (c) requires that any agent who was a partner of an agent outside A should be unmatched under $\hat{\mu}$. Consider the example illustrated in Figure 3, where all of group deviations from μ_0 by the group $A := \{m_1, w_1, w_2\}$ are illustrated. Consider a matching $\hat{\mu}_1$. Condition (a) is satisfied, since m_1 is matched with w_1 and w_2 is unmatched; condition (b) is satisfied, since m_3 and w_3 remain matched; condition (c) is satisfied, since m_2 who was matched with w_2 in A becomes unmatched.

Definition 5. Given a dynamic matching ϕ, a **(dynamic) group deviation** from ϕ is a pair $(A, \hat{\phi})$ consisting of a group A and a dynamic matching $\hat{\phi}$ such that there is a subset $\mathcal{H}'$ of $\mathcal{H}$,
(a) for each h in $\mathcal{H}'$, a pair $(A, \hat{\phi}(h))$ is a static group deviation from a matching $\phi(h)$, and
(b) for each h in $\mathcal{H} \setminus \mathcal{H}'$, $\hat{\phi}(h) = \phi(h)$.
Moreover, it is called **history-independent** if in each period, a matching inside the group A is history-independent, i.e., for each $t = 0, \cdots, T$, for each h^t and $\tilde{h}^t$ in $\mathcal{H}^t$, if h^t is in $\mathcal{H}'$,

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one considered by Roth and Vande Vate (1990) where a new matching $\hat{\mu}$ is obtained from μ by *satisfying* the blocking pair. The basic idea is also the same as Corbae, Temzelides and Wright (2003). In addition, this notion is different from *enforcement* used to define a bargaining set in Klijn and Massó (2003).

then $\tilde{h}^t$ is in $\mathcal{H}'$ and $\hat{\phi}(h^t)|_A = \hat{\phi}(\tilde{h}^t)|_A$.

In the dynamic group deviation $(A, \hat{\phi})$ from ϕ, at histories h in $\mathcal{H}'$ agents in A reorganize their match within A and the others remain matched at $\phi(h)$. In the remaining histories all agents are matched at $\phi(h)$ and possibly matched with agents outside A, which makes our dynamic group deviation different from deviations permitted in the core. In addition, if a dynamic group deviation is history-independent, the matching consisting only of agents in A is a function of calender time alone and need not be constant across periods. However, matchings of the agents outside A can be different across histories in a given period, so $\hat{\phi}$ need not be a history-independent dynamic matching. If the original dynamic matching ϕ is history-independent and a dynamic group deviation $(A, \hat{\phi})$ from ϕ is history-independent, then $\hat{\phi}$ is history-independent by the definition of static group deviation. The adjective "dynamic" is omitted when there is no confusion. For an example, consider a dynamic matching ϕ specifying μ_0 at each history in the repeated market of the constituent market depicted in Figure 3. One possible group deviation $\hat{\phi}$ by $\{m_1, w_1, w_2\}$

$$\begin{aligned}
\hat{\phi}(h) &= \mu_1 \quad \text{if } h = \emptyset, \\
&= \mu_2 \quad \text{if } h = \mu_3, \\
&= \mu_0 \quad \text{otherwise.}
\end{aligned}$$

For convenience, the group deviation is called **pairwise** if it consists either of an individual or of a pair of one man and one woman.

A group A is said to **block** the dynamic matching ϕ (via $\hat{\phi}$) if $(A, \hat{\phi})$ is a dynamic group deviation from ϕ and $U_i(\hat{\phi}) > U_i(\phi)$ for each i in A. Now we are ready to introduce our concept:[10]

Definition 6.

1. A dynamic matching ϕ is **(dynamically) group-stable** if no group blocks it; i.e., if there is no group deviation $(A, \hat{\phi})$ from ϕ such that $U_A(\hat{\phi}) > U_A(\phi)$.
2. A dynamic matching ϕ is **individually rational** if its outcome path is individually rational.

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[10] The term "group stability" used in many-to-one or many-to-many matching problems is different from ours, although we adopt the same approach of non-characteristic function. See the Related Literature section.

3. In the special case of a static market (i.e. $T = 0$), a matching μ is called **(statically) group-stable** if it is dynamically group-stable.
4. Moreover, if we consider only pairwise deviations, it is called **(dynamically) pairwise-stable**.

Note that a dynamic market with horizon $T = 0$ is a static market.

Lemma 1. *For a static market, the following are equivalent:*
(a) A matching is stable.
(b) It is in the core.
(c) It is statically group-stable.

For the equivalence of (a) and (b), see Theorem 3.3 in Roth and Sotomayor (1990). To show the equivalence of (b) and (c), observe that in both concepts only a deviating group matters but not the outsiders in a static setting.

Proposition 1. *If a dynamic matching is group stable, then its outcome path is in the core. The converse is not always true.*

The proof of the first part is in the Appendix. For the latter part, see Examples 2 and 3 in the next subsection. In addition, we may not have a group stable dynamic matching, as shown in the next subsection. However, if we restrict our attention to *repeated* markets, Proposition 2 can guarantee the existence of *pairwise* stable dynamic matching.

Proposition 2 (**Existence of a pairwise stable dynamic matching in "repeated" markets**)**.** *There exists a pairwise stable dynamic matching for each finitely or infinitely repeated market.*

Picking a stable matching in the constituent market, consider a dynamic matching assigning this stable matching everywhere. Individuals and a pair of a man and a woman cannot block this dynamic matching, since it assigns a stable matching everywhere and the constituent market is repeated. Thus, the dynamic matching is pairwise stable.

We make three remarks. First, we do not need strict preferences for this proposition to hold. Second, if a matching is not pairwise but group stable, there may be no group stable dynamic matching (cf. Example 2 in the next subsection). Finally, if we have a "dynamic" market, there may be no pairwise stable dynamic matching (cf. Example 3 in the next subsection.).

Before considering some examples, it is useful to characterize dynamic group stability. First, consider a dynamic market with finite horizon $(M, W, \{u_i^t\}_{i \in I, t=0,\cdots,T})$. At history $h^t \in \mathcal{H}$, the **sub-dynamic (marriage) market** is a dynamic market $(M, W, \{u_i^\tau\}_{i \in I, \tau=t,\cdots,T})$. Given a dynamic matching ϕ for the original market, define a **continuation dynamic matching** to be a function $\phi|_{h^t} : \mathcal{M}^{T-t+1} \rightarrow \mathcal{M}$ given by $\phi|_{h^t}(h^\tau) = \phi(h^t h^\tau)$ for each $h^\tau \in \mathcal{M}^{T-t+1}$.

Turning to the infinite horizon case ($T = \infty$), at history $h^t \in \mathcal{H}$, the **sub-dynamic (marriage) market** is $(M, W, \{u^\tau_{i \in I, \tau=t,\cdots,\infty}\})$. Given a dynamic matching ϕ for the original market, define a **continuation dynamic matching** to be a function $\phi|_{h^t} : \mathcal{H} \rightarrow \mathcal{M}$ given by $\phi|_{h^t} = \phi(h^t h^\tau)$ for each $h^\tau \in \mathcal{H}$. Now we are ready to state:

Lemma 2 (**Partial characterization of group stable dynamic matchings**). *Consider a dynamic market with finite or infinite horizon. If there is a group stable dynamic matching ϕ, then for each history h on the outcome path, the continuation dynamic matching $\phi|_h$ is group stable in the sub-dynamic market starting at h.*

The proof is straightforward and so we omit it.

3.2 Examples

Example 1 *(Continued).* The outcome path $\boldsymbol{\mu}_2 := (\mu_b, \mu_b)$ was in the core, and is supported by the following group stable dynamic matching:

$$\begin{aligned} \phi(h) &= \mu_a \quad \text{if } h = \mu_a, \\ &= \mu_b \quad \text{otherwise.} \end{aligned}$$

However, the dynamic matching specifying μ_b at each history cannot be group stable, as we discussed before. Thus, we need to consider history-dependent contingent plans. □

Example 2. (The core is nonempty but there is no group stable dynamic matching)

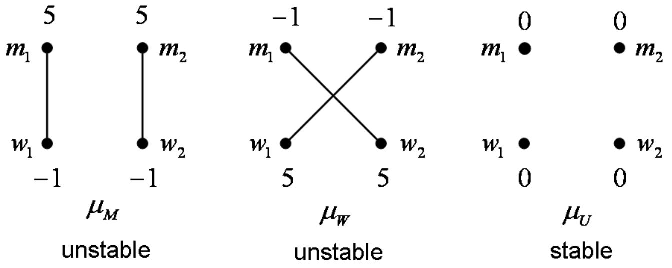

Figure 4: The constituent market in Example 2

Consider a twice repeated market with no discounting whose constituent market[11] is depicted in Figure 4. Here $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. There are seven possible matchings, but only three of them are depicted. Note that the matching μ_M is man-preferred but unstable, μ_W is woman-preferred but unstable, and μ_U is uniquely stable.

First, it can be verified that outcome paths (μ_M, μ_W) and (μ_W, μ_M) are in the core. Next, we show that there is no group stable dynamic matchng. Suppose for a contradiction that there is a group stable dynamic matching ϕ. Let $\{\mu^0, \mu^1\}$ be its outcome path. It follows from Lemma 2 that $\mu^1 = \mu_U$, since μ_U is a unique stable matching in the constituent market. There are two cases to consider. Suppose $\mu^0 \neq \mu_U$. Then, there exists at least one agent i who obtains the payoff of -1 in period 0. In total, his or her payoff is -1 under ϕ. All agents can be unmatched in both periods, which provides a return of 0. Thus, agent i blocks ϕ. This contradicts that ϕ is group stable. On the other hand, suppose $\mu^0 = \mu_U$. Since $\phi(\mu_U) = \mu_U$, each agent gets the payoff of 0. However, the group $I \equiv \{m_1, m_2, w_1, w_2\}$ can make a deviation ϕ' such that $\phi'(\emptyset) = \mu_M$ and $\phi'(\mu_M) = \mu_W$. Then, each agent's payoff is 4. So, the group I blocks ϕ via ϕ'. In any case, some group blocks ϕ. This is a contradiction. $\square$

Example 3. (The core is nonempty but there is no pairwise stable dynamic matching)

Consider a two-period dynamic market[12] depicted in Figure 5. Here, there are man m and woman w. Unlike the previous example, preferences vary across periods. In each period, the matching μ_U (unmatched) is stable, while the matching μ_T (together) is not stable. It can be verified that the outcome path (μ_T, μ_T) is in the core. Similarly to the previous example, we can show by contradiction that there is no pairwise stable dynamic matching.

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[11] This example is from Damiano and Lam (2005).

[12] This is adapted from an example in footnote 5 in Corbae, Temzelides and Wright (2003).

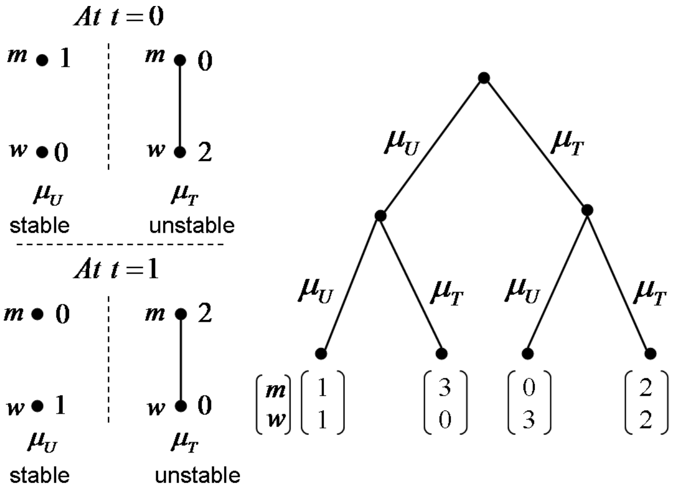

Figure 5: Constituent markets and total utilities in Example 3

□

4 Credible Group Stability

4.1 Definition

The question on the robustness of clearinghouses in the British medical markets which we raised in the Introduction can now be restated: What kind of stability concept supports a history-independent dynamic matching assigning a men-optimal stable matching in each period? We saw in the previous section that dynamic group stability does not always work. Remember that we consider all group deviations in the definition of dynamic group stability. Some of them may not be *defensible* in the sense that some members of the deviating group have an incentive to reorganize their match inside or outside the group which makes all of the agents strictly better off. We develop the concept of *defensibility*, and then that of *credible group stability* as immunity against defensible group deviations.

Example 4. Consider a two-period dynamic market with $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Constituent markets are illustrated in Figure 6, where the utilities of being unmatched for all agents are 0 in both markets. The payoffs depending on outcome paths

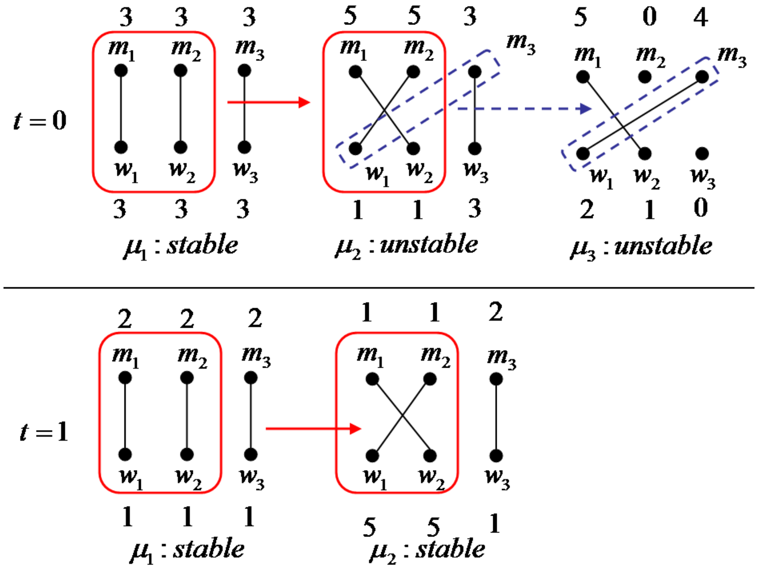

Figure 6: The preferences in the constituent market in Example 4

are calculated in Figure 7. Note that the figures do not include all possible matchings. In period 0 market there is a unique stable matching μ_1,[13] while in period 1 market μ_1 and μ_2 are men-optimal and women-optimal stable matchings, respectively.

In this market, the dynamic matching ϕ specifying the men-optimal stable matching μ_1 in both periods is not group stable, because the group $A := \{m_1, m_2, w_1, w_2\}$ blocks it via the history-independent dynamic matching $\hat{\phi}$ which specifies μ_2 everywhere. This is illustrated in Figures 6 and 7 by thick circles and arrows.

Consider the possibility of further deviations for the group deviation $(A, \hat{\phi})$. As we can see, no matter how the group A reorganizes its match inside the group, no agent in A can be better off. Note that if we restrict the market to the group A, $\mu_2|_A$ is a men-optimal stable matching in period 0 and women-optimal stable matchings in period 2, although it is not even stable in the original market for period 0. The coordination of men-optimal and women-optimal stable matchings in the restricted markets makes all agents in A better off and has no further deviation within the group. In this sense, the group deviation $(A, \hat{\phi})$ is *credible*, and so the dynamic matching ϕ is not immune to such credible group deviations.

However, closely looking at the group deviation, we notice that woman w_1 in A can be

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[13] We implicitly assume that $u^0_{m_2}(w_3) < u^0_{m_2}(w_2)$ so that we have a unique stable matching.

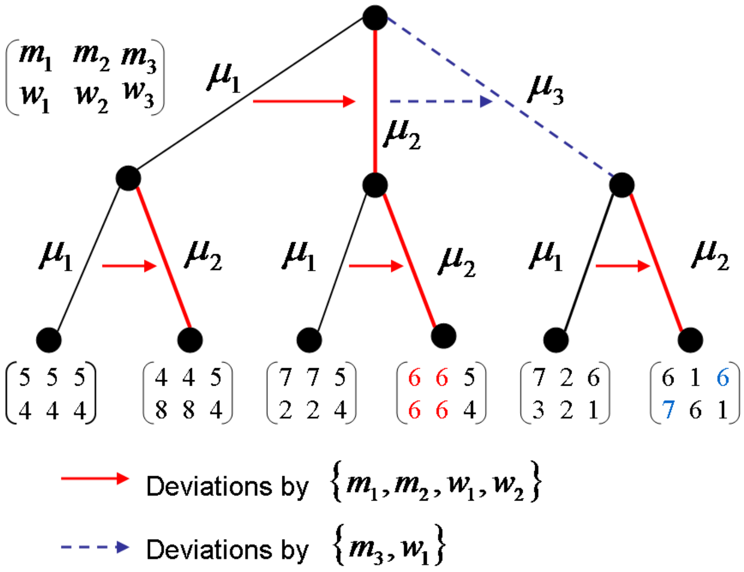

Figure 7: The total utilities in Example 4: The thick arrows stand for a deviation by $\{m_1, m_2, w_1, w_2\}$, and the dotted arrow indicates further deviation by $\{m_3, w_1\}$

better off with m_3 in period 0 who is "outside" the deviating group A, keeping the matching at period 1 fixed (This is illustrated in Figures 6 and 7 by dotted circles and arrows). That is, the group $\{m_3, w_1\}$ blocks $\hat{\phi}$ via the history-independent dynamic matching $\bar{\phi}$ which specifies μ_3 and μ_2 in periods 0 and 1, respectively. We say that a group deviation is *defensible* if some members of the group have no further profitable deviation by matching with agents inside or outside the group. Although it is *credible* in the sense of the previous paragraph, the group deviation $(A, \hat{\phi})$ is not defensible. We consider only defensible deviations in the solution concept which we define next. □

With this example in mind, we formalize the concept.

Definition 7. Given a dynamic matching ϕ, a group deviation $(A, \hat{\phi})$ from ϕ is **defensible** if

(a) it is history-independent, and

(b) there is no group deviation $(B, \bar{\phi})$ from $\hat{\phi}$ with $A \cap B \neq \emptyset$ such that $U_i(\bar{\phi}) > U_i(\hat{\phi})$ for each i in B.[14]

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[14] Even if we require the group deviation $(B, \bar{\phi})$ to be history-independent, all of our results are not affected. In this case, since the set of the modified defensible group deviations is larger than that of the original one,

Any members in a defensible group deviation cannot reorganize their match inside or outside the group via a history-independent group deviation which makes all agents strictly better off. It may seem strong to require a defensible group deviation to be history-independent, but this condition would be acceptable if we consider complexity of contingent plans. Using this defensibility, we introduce the notion of credible group stability:

Definition 8. *1.* A dynamic matching ϕ is **credibly group-stable** if it is individually rational, and there is no defensible group deviation $(A, \hat{\phi})$ such that $U_i(\hat{\phi}) > U_i(\phi)$ for each i in A.

2. If A is pairwise in the above definition, the the credible group stability is called **credible pairwise stability**.

In other words, a credibly group-stable dynamic matching is individually rational and immune against profitable and defensible group deviations. In a static market, our credible *pairwise* stability coincides with *weak stability*[15] introduced by Klijn and Massó (2003). The idea of our credible group stability is similar to the bargaining set.[16]

Lemma 3. *In a static market,*
(a) a stable matching is credibly group-stable,
(b) a credibly group-stable matching is not always stable, and
(c) a credibly pairwise-stable matching is not always credibly group-stable.

The first statement is obvious, since a stable matching is group stable by Lemma 1. For the rest, examples are given in the Appendix. Hence, credible group stability is strictly stronger than credible pairwise stability, and strictly weaker than stability.

The following proposition is the key in proving the existence of credible group stability for dynamic markets. The proof is in the Appendix.

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the set of credibly group-stable dynamic matchings that use the modified defensibility is smaller than that of the original one.

[15] See Definition 10 and Proposition 5 in the Appendix for the definition and the proof, respectively.

[16] The definition of our group deviation is different from that of enforcement which is used to define the Zhou's bargaining set as formalized by Klijn and Massó (2003) for a marriage model, and thus there is no obvious relationship between our credible group stability and the bargaining set. However, Klijn and Massó (2003) that the set of weakly stable and weakly efficient matchings coincides with the bargaining set. Hence, the set of credible pairwise stable and weakly efficient matchings coincides with the bargaining set.

Proposition 3. *In a static market, for each stable matching μ, if a group deviation $(A, \hat{\mu})$ from μ is defensible, then $\hat{\mu}$ is stable.*

4.2 Existence

Theorem 2 (Existence). *For every dynamic market with either finite or infinite horizon, there exists a credibly group-stable dynamic matching.*

Consider any dynamic market with either finite or infinite horizon. From Theorem 1, there exist a men-optimal stable matching and a women-optimal stable matching in each period market. Then, we have either a history-independent dynamic matching assigning a men-optimal stable matching in each period or the one assigning a women-optimal stable matching in each period. Theorem 2 follows by showing that both are credibly group-stable:

Proposition 4. *In a dynamic market with finite or infinite horizon, a history-independent dynamic matching assigning a men-optimal stable matching in each period is credibly group-stable. Similarly, the result holds for a women-optimal stable matching.*

Proof. Pick a men-optimal stable matching μ_M^t in each period t market. Let ϕ be a history-independent dynamic matching with $\phi(h^t) = \mu_M^t$ for each h^t in $\mathcal{H}$. We show that ϕ is credibly group-stable. First, since each μ_M^t is individually rational in the corresponding period market, ϕ is individually rational. Next, fix a defensible group deviation $(A, \hat{\phi})$ from ϕ. Denote the outcome path of $\hat{\phi}$ by $(\hat{\mu}^0, \hat{\mu}^1, \cdots, \hat{\mu}^T)$. We need to show that $U_i(\phi) \geq U_i(\hat{\phi})$ for some i in A.

Note that from the definition of dynamic group deviation, for each $t = 0, \cdots, T$,

$$\text{either } \hat{\mu}^t = \mu_M^t \text{ or } (A, \hat{\mu}^t) \text{ is a static group deviation from } \mu_M^t. \tag{1}$$

There are two cases to consider: First, consider the case where some man m is in A.
Step 1: Show that for each $t = 0, \cdots, T$, either $\hat{\mu}^t = \mu_M^t$, or $\hat{\mu}^t$ is stable in period t market. Suppose for a contradiction that for some period t, $\hat{\mu}^t \neq \mu_M^t$ and $\hat{\mu}^t$ is not stable. Then, it follows from (1) that $(A, \hat{\mu}^t)$ is a static group deviation from μ_M^t. By Proposition 3, $(A, \hat{\mu}^t)$ is not defensible. Thus, there exists a static group deviation $(B, \bar{\mu}^t)$ from $\hat{\mu}^t$ with $A \cap B \neq \emptyset$ such that $u_i^t(\bar{\mu}^t) > u_i^t(\hat{\mu}^t)$ for each i in B. Consider the following history-independent dynamic

matching:

$$\begin{aligned}\bar{\phi}(h^\tau) &= \hat{\mu}^\tau \quad \text{if } \tau \neq t,\\ &= \bar{\mu}^t \quad \text{if } \tau = t.\end{aligned}$$

Since dynamic matching ϕ is history-independent and dynamic group deviation $(A, \hat{\phi})$ is also history-independent, the dynamic matching $\hat{\phi}$ is history-independent. This implies that $(B, \bar{\phi})$ is a dynamic group deviation from $\hat{\phi}$. Then, the outcome path of $\hat{\phi}$ is $(\hat{\mu}^0, \cdots, \hat{\mu}^T)$, while the outcome path of $\bar{\phi}$ is $(\hat{\mu}^0, \cdots, \hat{\mu}^{t-1}, \bar{\mu}^t, \hat{\mu}^{t+1}, \cdots, \hat{\mu}^T)$. Thus,

$$U_i(\bar{\phi}) = \sum_{\tau \neq t} u_i^\tau(\hat{\mu}^\tau) + u_i^\tau(\bar{\mu}^t) > \sum_{\tau=0}^{T} u_i^\tau(\hat{\mu}^\tau) = U_i(\hat{\phi}) \text{ for each } i \text{ in } B.$$

This contradicts the assumption that the group deviation $(A, \hat{\phi})$ from ϕ is defensible. This completes the proof of Step 1.

Step 2: Show $U_m(\phi) \geq U_m(\hat{\phi})$. Since μ_M^t is a men-optimal stable matching in period t market, it follws from Step 1 that

$$\text{either } u_m^t(\mu_M^t) = u_m^t(\hat{\mu}^t) \text{ or } u_m^t(\mu_M^t) \geq u_m^t(\hat{\mu}^t).$$

This implies

$$U_m(\phi) \equiv \sum_{t=0}^{T} u_m^t(\mu_M^t) \geq \sum_{t=0}^{T} u_m^t(\hat{\mu}^t) \equiv U_m(\hat{\phi}).$$

This completes the proof of Step 2.

Next, consider the case where A consists only of women. Fix $w \in A$ and period t. Then, if $(A, \hat{\mu}^t)$ is a static group deviation from μ_M^t, all women in A are unmatched. Thus, from (1), either $\hat{\mu}^t = \mu_M^t$ or w is unmatched at $\hat{\mu}^t$. Since μ_M^t is individually rational in the period t market, either $u_w^t(\mu_M^t) = u_w^t(\hat{\mu}^t)$ or $u_w^t(\mu_M^t) \geq u_w^t(w) \equiv u_w^t(\hat{\mu}^t)$. Thus, $U_w(\phi) \geq U_w(\hat{\phi})$.

Therefore, we proved that for each defensible group deviation $(A, \hat{\phi})$ from ϕ, $U_i(\phi) \geq U_i(\hat{\phi})$ for some i in A. Hence, ϕ is credibly group-stable. □

4.3 Policy implications

Because a men-optimal (women-optimal) stable matching is favorable to men (women) but not to women (men), we can think of two compromises in market design. The first is a mechanism that always selects men-optimal and women-optimal stable matchings alternately. The second is a mechanism that always selects a median stable matching in each period which is neither men-optimal stable nor women-optimal stable. The question is: Is such a dynamic matching always credibly group-stable? The following example indicates that it is not.

Example 5. Consider a two-period dynamic market whose constituent markets are depicted in Figure 8, where utility values of being unmatched are 0. μ_M and μ_W indicate men-optimal and women-optimal stable matchings, respectively. In addition, μ_S denotes another stable matching in Figure 8.

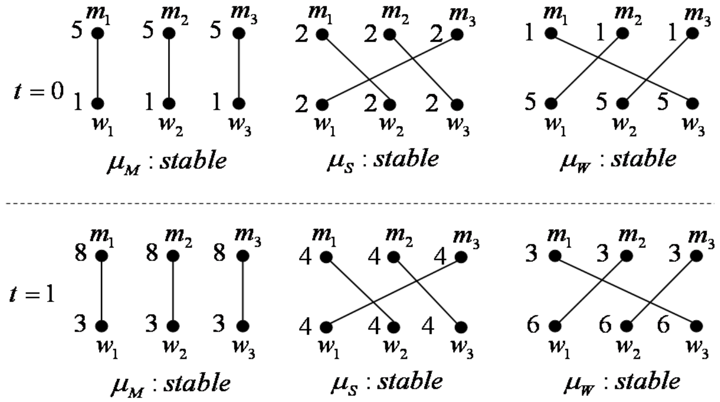

Figure 8: Constituent markets in Example 5

Case 1: A history-independent dynamic matching ϕ consisting only of μ_S is not credibly group-stable.

Consider a history-independent group deviation $(A, \hat{\phi})$ from ϕ where $A = M \cup W$, and $\hat{\phi}$ assigns μ_W and μ_M to period 0 and 1, respectively. All agents in A are better off in $\hat{\phi}$ than in ϕ. We show that $(A, \hat{\phi})$ is defensible. Suppose for a contradiction that there is a group deviation $(B, \bar{\phi})$ from $\hat{\phi}$ with $B \cap A = B \neq \emptyset$ such that $U_i(\bar{\phi}) > U_i(\hat{\phi})$ for each i in B. Note that each man obtains the payoff of 9 and each woman obtains that of 8 at $\hat{\phi}$, and no agent in B is not unmatched at $\bar{\phi}$ in each period. So, B includes matched pairs at $\bar{\phi}$. We have three cases to consider: First, if m_1 is in B, since he has the payoff more than 9 at $\bar{\phi}$, he is either matched with w_1 in both periods, or matched with w_2 and w_1 in periods 0 and 1, respectively. In the former case, w_1 is in B but gets the payoff of 4 at $\bar{\phi}$ that is less than 8. A contradiction. In the latter case, w_2 is in B, and cannot get the payoff more than 8 at $\bar{\phi}$.

A contradiction. Similarly, we can obtain a contradiction for the other two cases where m_2 is in B or m_3 is in B. Thus, $(A, \bar{\phi})$ is defensible. Thus, ϕ is not credibly group-stable.

Case 2: A history-independent dynamic matching ϕ consisting of μ_M in the first period and μ_W in the second period is not credibly group-stable.

Consider the same group deviation $(A, \hat{\phi})$ from ϕ as Case 1. All agents in A are better off in $\hat{\phi}$ than ϕ. Since $(A, \hat{\phi})$ is defensible as we verify in Case 1, ϕ is not credibly group-stable.

$\square$

5 Pareto Efficiency in Finitely Repeated Markets

In a static market, since any stable matching is in the core from Lemma 1, it is weakly Pareto efficient. Thus, the question of welfare does not arise in a static stable mechanism. However, as we saw in Example 2, a history-independent dynamic matching assigning a unique statically stable matching in each period is credibly group-stable, but not Pareto efficient even in a finitely repeated market. In this section, we investigate Pareto efficiency in finitely repeated markets. Whether an outcome path consisting of stable matchings is Pareto efficient depends on preferences of agents in constituent markets. To examine Pareto efficiency, we introduce a condition, called *regularity*, for a static market.

5.1 Regularity condition for static markets

To introduce the regularity condition, we define a **restricted market** $(\tilde{M}, \tilde{W}, \tilde{u})$, denoted by $(\tilde{M} \cup \tilde{W})$, of a static market (M, W, u) to be a static market such that $\tilde{M} \subset M$, $\tilde{W} \subset W$, $\tilde{u}_m$ is a restriction of u_m to $\tilde{W} \cup \{m\}$ for each $m \in \tilde{M}$, and $\tilde{u}_w$ is a restriction of u_w to $\tilde{M} \cup \{w\}$ for each $w \in \tilde{W}$. Moreover, throughout this section, a **pair** (i, j) means that either i belong to the opposite sex of j or $i = j$.

Definition 9. Given a matching μ with the number N of pairs formed in μ, a static market has **regularity** for μ if there is a sequence $\{(i_k, \mu(i_k))\}_{k=1}^{N}$ of pairs formed in μ (called a **regular sequence** for μ) such that
(a) for $k = 1$, i_1's most preferred mate is $\mu(i_1)$ in a restricted market $M \cup W$,
(b) for $k \geq 2$, i_k's most preferred mate is $\mu(i_k)$ in a restricted market $(M \cup W) \setminus \{i_l, \mu(i_l)\}_{l=1}^{k-1}$.

In a regular sequence $\{(i_k, \mu(i_k))\}_{k=1}^{N}$ for μ, agent i_1's partner at μ is $\mu(i_1)$ who is the best partner to i_1 among all agents. Removing this pair $(i_1, \mu(i_1))$ from the market, agent i_2's partner at μ is $\mu(i_2)$ who is the best partner to i_2 among all agents except the pair $(i_1, \mu(i_1))$. Removing the pairs $(i_1, \mu(i_1))$ and $(i_2, \mu(i_2))$ from the market, we repeat the same procedure until no agent is left.

As an example, consider the Example 2. The constituent market has regularity for μ_M, μ_W but not for μ_S. As a regular sequence for μ_M, take $i_1 = m_1$ and $i_2 = m_2$.

Lemma 4. *(1) If a static market has regularity for a matching μ, then μ may not be stable. (2) If a matching is stable in a static market, then the market may not have regularity for it.*

In the constituent market of Example 2, the matching μ_M satisfies regularity, but is not stable. On the other hand, the matching μ_U is stable but does not satisfy regularity. In a special class of markets with acceptability and $|M| = |W|$, the regularity condition is clearly equivalent to a sufficient condition for a unique stable matching identified by Eeckhout (2000). Thus, in this class, if a static market has regularity for a matching μ, then μ is uniquely stable.

5.2 Finitely Repeated markets

An outcome path $\boldsymbol{\mu}$ is **Pareto efficient** if there is no other outcome path $\boldsymbol{\mu}'$ such that $U_i(\boldsymbol{\mu}') \geq U_i(\boldsymbol{\mu})$ for each i in I with strict inequality for some i in I.

Theorem 3 (**Pareto efficiency**). *In a finitely repeated market, if a matching μ^* satisfies regularity in the constituent market, then an outcome path consisting of the matching μ^* is Pareto efficient.*

Proof. Let $(M, W, \{u_i\}_{i \in I})$ be a constituent market. Let the outcome path $\boldsymbol{\mu}^* := (\mu^*, \mu^*, \cdots, \mu^*)$. Take any outcome path $\boldsymbol{\mu} := (\mu^t)_{t=0}^{T}$ that is different from $\boldsymbol{\mu}^*$. We show that there exists an agent $i \in I$ such that $U_i(\boldsymbol{\mu}^*) > U_i(\boldsymbol{\mu})$.

Take a regular sequence $\{i_k, \mu^*(i_k)\}_{k=1}^{N}$ of pairs for μ^*. Take $\mathcal{M}(i) := \{\mu \in \mathcal{M} | (i, \mu^*(i)) \notin \mu\}$. We choose a particular agent i_K among $\{i_k\}_{k=1}^{N}$ in the following way:
Step 1: If there exists $t = 0, \cdots, T$ such that $\mu^t \in \mathcal{M}(i_1)$, then set $i_K = i_1$. Otherwise, go to the next step.

Step k: If there exists $t = 0, \cdots, T$ such that $\mu^t \in \mathcal{M}(i_k)$, then set $i_K = i_k$. Otherwise, go to the next step.

This procedure stops after at most N steps. In addition, we can choose such an agent i_K. Otherwise, we would have a contradiction that $\boldsymbol{\mu}^* = \boldsymbol{\mu}$.

To show that $U_{i_K}(\boldsymbol{\mu}^*) > U_{i_K}(\boldsymbol{\mu})$, it is sufficient to show that for each $\mu^t \in \mathcal{M}(i_K)$, $u_{i_K}(\mu^*) > u_{i_K}(\mu^t)$. Note that for each $\mu^t \in \mathcal{M} \setminus \mathcal{M}(i_K)$, $u_{i_K}(\mu^*) = u_{i_K}(\mu^t)$.

Fix $\mu^t \in \mathcal{M}(i_K)$, i.e., $(i_K, \mu^*(i_K)) \notin \mu^t$. Because of the procedure of finding i_K, agents in $\{i_k, \mu^*(i_k)\}_{k=1}^{K-1}$ are matched with each other, and thus agent i_K is not matched with any mate in $\{i_k, \mu^*(i_k)\}_{k=1}^{K-1}$. By regularity and strict preferences, agent i_K's most preferred mate in the restricted market $(M \cup W) \setminus \{i_k, \mu^*(i_k)\}_{k=1}^{K-1}$ is $\mu^*(i_K)$, and thus $u_{i_K}(\mu^*) > u_{i_K}(\mu^t)$.

□

Corollary 1. *In a finitely repeated market, if a stable matching μ^* satisfies regularity, then an outcome path consisting the matching μ^* is Pareto efficient.*

Any outcome path consisting of the men-optimal (or women-optimal) stable matching of the constituent market can be supported via credible group stability by Theorem 2. However, in the Example 2, such an outcome path consisting only of μ_U is not Pareto efficient, and the regularity condition is not satisfied. Together with the following example, a partial converse[17] of the Corollary 1 may hold:

Example 6. (An example for the partial converse of Corollary 1.) Consider a three-times repeated market with $M = \{m_1, m_2, m_3, m_4\}$, $W = \{w_1, w_2, w_3, w_4\}$ and the following preferences[18] with no discounting:

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[17] Conjecture of a partial converse: If a static market does not have regularity for a stable matching μ, there is a period T such that in the T times repeated market, an outcome path consisting of μ is not Pareto efficient.

[18] This example for ordinal preferences is from Example 2 in Eeckhout(2000), which Ahmet Alkan suggests. We attach utility values so that the claim holds.

m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4
w_3 (6)	w_4 (6)	w_1 (6)	w_3 (6)	m_2 (6)	m_1 (6)	m_2 (6)	m_3 (6)
w_1 (2)	w_2 (2)	w_3 (2)	w_4 (2)	m_1 (2)	m_2 (2)	m_3 (2)	m_4 (2)
w_2 (1)	w_3 (1)	w_2 (1)	w_2 (1)	m_3 (1)	m_3 (1)	m_4 (1)	m_1 (1)
w_4 (0)	w_1 (0)	w_4 (0)	w_1 (0)	m_4 (0)	m_4 (0)	m_1 (0)	m_2 (0)

The numbers in parentheses indicate utilities. Each agent is acceptable to all those of the opposite sex. Note that there is a unique stable matching $\mu^* = \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\}$ and the constituent market does not have regularity for μ^*. Consider three matchings $\mu^0 = \{(m_1, w_2), (m_2, w_1), (m_3, w_4), (m_4, w_3)\}$, $\mu^1 = \{(m_1, w_3), (m_2, w_4), (m_3, w_1), (m_4, w_4)\}$ and $\mu^2 = \{(m_1, w_2), (m_2, w_3), (m_3, w_1), (m_4, w_4)\}$. Then, total utilities are calculated as follows:

Total utilities	m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4
$U_i(\mu^*, \mu^*, \mu^*)$	6	6	6	6	6	6	6	6
$U_i(\mu^0, \mu^1, \mu^2)$	8	7	12	10	8	14	7	8

Thus, the outcome path consisting only of μ^* is not Pareto efficient.

□

6 Conclusion

Some real-life dynamic matching markets use a mechanism that finds a men-optimal or a women-optimal stable matching. Our result shows that this approach does not create instability in a dynamic setting. Therefore, this approach is justified.

Appendix - Examples and proofs

An example of empty core. Consider a two-period dynamic market with $M = \{m_1, m_2\}$, $W = \{w_1, w_2, w_3\}$. The preferences are depicted in Figure 9. In addition, the utility of being unmatched is 0 to each agent. Note that the Figure 9 just indicates the preferences for all agents, but does not show all matchings. There are eleven possible matchings. Denote μ_{ij} by the matching in which m_i is matched with w_j and the other agents are unmatched.

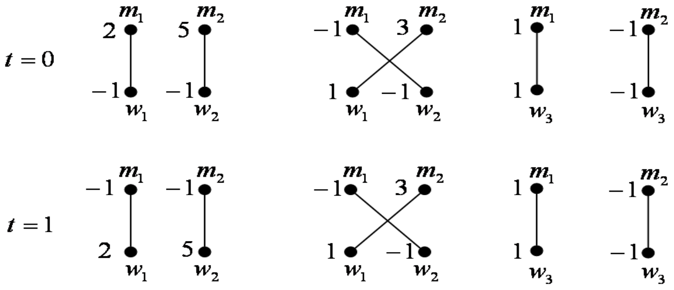

Figure 9: The preferences in the constituent market

Denote $\mu_{ij,kl}$ by the matching in which m_i (m_k) is matched with w_j (w_l) and the other agent is unmatched. μ_U is the matching where all agents are unmatched. In total, we have $121 = 11 \times 11$ outcome paths. Out of them, we have 15 individual rational outcome paths: (μ_{11}, μ_{11}), (μ_{11}, μ_{21}), (μ_{13}, μ_{11}), (μ_{13}, μ_{13}), (μ_{13}, μ_{21}), (μ_{13}, μ_U), (μ_{21}, μ_{13}), (μ_{21}, μ_U), (μ_U, μ_{13}), (μ_U, μ_{21}), (μ_U, μ_U); (μ_{21}, μ_{21}), (μ_{21}, μ_{22}), (μ_{22}, μ_{22}),$(\mu_{11,22}, \mu_{11,22})$. The first eleven outcome paths are blocked by the pair (m_2, w_2) via (μ_{22}, μ_{22}), and the last four are blocked by (m_1, w_3) via (μ_{13}, μ_{13}). □

Proof of Proposition 1. Let a dynamic matching ϕ be group stable. Suppose for a contradiction that its outcome path $\boldsymbol{\mu}(\phi) = \{\mu^t(\phi)\}_{t=0}^T$ is not in the core. Then, there exist a group A and an outcome path $\hat{\boldsymbol{\mu}} := \{\hat{\mu}^t\}_{t=0}^T$ such that $U_i(\hat{\boldsymbol{\mu}}) > U_i(\boldsymbol{\mu}(\phi))$ for each i in A. Then, for each t take a matching $\tilde{\mu}^t$ such that $(A, \tilde{\mu}^t)$ is a static group deviation from $\mu^t(\phi)$ and $\tilde{\mu}^t(i) = \hat{\mu}^t(i)$ for each i in A. Consider the dynamic group deviation $(A, \tilde{\phi})$:

$$\begin{aligned} \tilde{\phi}(h) &= \tilde{\mu}^t \quad \text{if } h = (\hat{\mu}^0, \cdots, \hat{\mu}^{t-1})), \\ &= \phi(h) \quad \text{otherwise.} \end{aligned}$$

Then, $U_i(\tilde{\phi}) > U_i(\phi)$ for each i in A. A contradiction. □

Proof of equivalence between credible pairwise stability and weak stability.

Proposition 5. *In a static market, a matching is credibly pairwise-stable if and only if it is weakly stable.*

Definition 10 (Klijn and Massó, 2003). Consider a static market.

1. A blocking pair (m, w) for μ is **weak** if there is a woman $w' \in W$ such that $u_m(w') > u_m(w)$ and (m, w') is a blocking pair for μ, or a man $m' \in M$ such that $u_w(m') > u_w(m)$ and (m', w) is a blocking pair for μ. Here, (m, w) is a blocking pair for μ if the pair blocks μ.
2. A matching μ is **weakly stable** if it is individually rational and all blocking pairs are weak.

To prove the equivalence, we show that if a matching μ is individually rational,

$$\begin{aligned} & \text{all blocking pairs for } \mu \text{ are weak} \\ \Leftrightarrow \quad & \text{there is no pairwise deviation } (A, \hat{\mu}) \text{ from } \mu,\ u_i(\hat{\mu}) > u_i(\mu) \text{ for each } i \text{ in } A \\ \Leftrightarrow \quad & \text{for each pairwise deviation } (A, \hat{\mu}) \text{ from } \mu, \text{ if } u_i(\hat{\mu}) > u_i(\mu) \text{ for each } i \text{ in } A, \\ & \quad \text{then } (A, \hat{\mu}) \text{ is not defensible.} \end{aligned}$$

The equivalence of the second and the third statements is a logical consequence. We show the equivalence of the first and the third statements. Suppose that μ is individually rational.

First, we show the direction $(\Rightarrow)$. Suppose that all blocking pairs for μ is weak. Let $(A, \hat{\mu})$ be a pairwise deviation from μ such that $u_i(\hat{\mu}) > u_i(\mu)$ for each i in A. Without loss of generality, take $m \in A$. Then, since m is in A and μ is individually rational, $u_m(\hat{\mu}) > u_m(\mu) \geq u_m(m)$. This implies that m is matched with some woman in A at $\hat{\mu}$. Denote this woman by w. Then, the pair (m, w) blocks μ. We show that a pairwise deviation $(A, \hat{\mu}) = (\{m, w\}, \hat{\mu})$ is not defensible. Since all blocking pairs are weak by our hypothesis, without loss of generality,

$$\exists w' \in W, u_m(w') > u_m(w), \text{and} \tag{2}$$

$$(m, w') \text{ is a blocking pair for } \mu. \tag{3}$$

By the definition of pairwise deviation, either $w' \in A$, w' is unmatched at $\hat{\mu}$, or w is matched with $\mu(w)$ at $\hat{\mu}$. If w' were in A, $w' \neq w$ by (5), which would contradict that $A = \{m, w\}$. If w' were unmatched at $\hat{\mu}$, then $(m, w') \in \mu$ from the definition of pairwise deviation, and thus would contradict (3). Thus, w is matched with $\mu(w)$ at $\hat{\mu}$. Now, we consider a pairwise deviation $(\{m, w'\}, \bar{\mu})$ with $(m, w') \in \bar{\mu}$. Then, it follows from (3) that $u_{w'}(m) \equiv u_{w'}(\bar{\mu}) > u_{w'}(\mu) \equiv u_w(\hat{\mu})$. Moreover, it follows from (5) that $u_m(w') \equiv u_m(\bar{\mu}) > u_m(w) \equiv u_m(\hat{\mu})$. Thus, the pairwise deviation $(A, \hat{\mu})$ is not defensible.

Next, we show the other direction $(\Leftarrow)$. Suppose that the hypothesis is true. Let (m, w) be a blocking pair of μ. Then, consider the pairwise deviation $(\{m, w\}, \hat{\mu})$ from μ with $(m, w) \in \hat{\mu}$.

Then, $u_m(\hat{\mu}) > u_m(\mu)$ and $u_w(\hat{\mu}) > u_w(\mu)$. By our hypothesis, the pairwise deviation is not defensible. Thus, there is a group deviation $(B, \bar{\mu})$ from $\hat{\mu}$ with $\{m, w\} \cap B \neq \emptyset$ such that $u_i(\bar{\mu}) > u_i(\hat{\mu})$ for each i in B. Without loss of generality, take m in $\{m, w\} \cap B$. Then,

$$u_m(\bar{\mu}) > u_m(\hat{\mu}) > u_m(\mu) \geq u_m(m). \tag{4}$$

The last inequality follows from individual rationality of μ. The inequalities (4) imply that m is matched with some woman at $\bar{\mu}$ who is in B. Denote this woman by w'. We show that the pair (m, w') is a blocking pair for μ. Since $(m, w') \in \bar{\mu}$, the inequalities (4) imply that $u_m(w') > u_m(\mu)$. Now, it is sufficient to show $u_{w'}(m) > u_w(\mu)$. By the definition of pairwise deviation, either $w' = w$, w' is unmatched at $\hat{\mu}$, or w' is matched with $\mu(w')$ at $\hat{\mu}$. However, $w' \neq w$, since we have (4), $(m, w) \in \hat{\mu}$ and $(m, w') \in \bar{\mu}$. Moreover, if she were unmatched at $\hat{\mu}$, then it would follow from the definition of pairwise deviation that w' is matched with m at $\hat{\mu}$, that is, we would have (m, w') in μ and $\bar{\mu}$, contradicting the inequalities (4). Hence, w' is matched with $\mu(w')$ at $\hat{\mu}$ and thus $u_{w'}(m) \equiv u_{w'}(\bar{\mu}) > u_{w'}(\hat{\mu}) = u_{w'}(\mu)$. The inequality holds because w' is in B.

□

Lemma 3 (b). Consider a static market (Knuth, 1976) with $M = \{m_1, m_2, m_3, m_4\}$, $W = \{w_1, w_2, w_3, w_4\}$ and the following preferences:

m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4
$\boldsymbol{w_1}$	w_2	w_3	$\boldsymbol{w_4}$	m_4	$\boldsymbol{m_3}$	$\boldsymbol{m_2}$	m_1
w_2	w_1	w_4	w_3	m_3	m_4	m_1	m_2
w_3	w_4	w_1	w_2	m_2	m_1	m_4	m_3
w_4	$\boldsymbol{w_3}$	$\boldsymbol{w_2}$	w_1	$\boldsymbol{m_1}$	m_2	m_3	$\boldsymbol{m_4}$

where each column indicates the preference of an agent in the first row, all mates are acceptable in each column, and an upper mate is strictly preferred to the lower one. Consider the matching $\mu := \{(m_1, w_1), (m_2, w_3), (m_3, w_2), (m_4, w_4)\}$. Each of boldfaced cells in the table indicates his or her partner from this matching. This matching is not stable (for example, a pair (m_2, w_1) blocks it) but individually rational. We show by contradiction that μ is credibly group-stable. Suppose for a contradiction that there exists a defensible group deviation $(A, \hat{\mu})$ such that $u_A(\hat{\mu}) > u_A(\mu)$. Note that A does not contain the agents m_1, m_4, w_2 nor w_3, because m_1, m_4, w_2 and w_3 have the best mate in the matching μ.

First, consider the case where A is a pair. Then, since A blocks μ, A is (m_2, w_1), (m_3, w_1), (m_2, w_4), or (m_3, w_4). If $A = (m_2, w_1)$, then the pair (m_3, w_1) blocks $\hat{\mu}$. If $A = (m_3, w_1)$, then the pair (m_3, w_4) blocks $\hat{\mu}$. If $A = (m_2, w_4)$, then (m_2, w_1) blocks $\hat{\mu}$. Finally, if $A = (m_3, w_4)$, then (m_2, w_4) blocks $\hat{\mu}$. Hence, the deviation $(A, \hat{\mu})$ is not defensible. A contradiction.

If A consists of three agents, it is not defensible since the deviation is similar to pairwise ones. A contradiction. Thus, $A = \{m_2, m_3, w_1, w_4\}$. By the defensibility, the restriction $\hat{\mu}|_A$ to A is stable in the restricted market consisting of A. This implies that $\hat{\mu}|_A$ is either $\{(m_2, w_1), (m_3, w_4)\}$ or $\{(m_2, w_4), (m_3, w_1)\}$. In both cases, since m_4 is unmatched at $\hat{\mu}$, (m_4, w_1) blocks $\hat{\mu}$. A contradiction. $\square$

Lemma 3 (c) . Consider a static market with $M = \{m_1, m_2, m_3, m_4\}$, $W = \{w_1, w_2, w_3, w_4\}$, and the following preferences:

m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4
$\boldsymbol{w_1}$	w_1	w_4	w_1	m_3	$\boldsymbol{m_2}$	$\boldsymbol{m_3}$	m_2
	w_4	w_1	$\boldsymbol{w_4}$	m_4			m_3
	$\boldsymbol{w_2}$	$\boldsymbol{w_3}$		m_2			$\boldsymbol{m_4}$
				$\boldsymbol{m_1}$			

where each column indicates the preference of an agent in the first row, only acceptable mates are listed in each column, and an upper mate is strictly preferred to the lower one. Consider the matching $\mu = \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\}$. Each of boldfaced cells in the table indicates his or her partner from this matching. This matching is not stable but individually rational. All of blocking pairs are (m_2, w_1), (m_3, w_1), (m_4, w_1), (m_2, w_4), and (m_3, w_4).

First, we show that μ is credibly pairwise-stable. Suppose for a contradiction that there is a defensible pairwise deviation $(A, \hat{\mu})$ from μ such that $u_A(\hat{\mu}) > u_A(\mu)$. Then, since A is a blocking pair, A is (m_2, w_1), (m_3, w_1), (m_4, w_1), (m_2, w_4), or (m_3, w_4). If $A = (m_2, w_1)$, then the pair (m_3, w_1) blocks $\hat{\mu}$. If $A = (m_3, w_1)$, then the pair (m_3, w_4) blocks $\hat{\mu}$. If $A = (m_4, w_1)$, then the pair (m_3, w_1) blocks $\hat{\mu}$. If $A = (m_2, w_4)$, then the pair (m_2, w_1) blocks $\hat{\mu}$. Finally, if $A = (m_3, w_4)$, then the pair (m_2, w_4) blocks $\hat{\mu}$. Thus, we have a contradiction: the pair deviation $(A, \hat{\mu})$ is not defensible. Hence, μ is credibly pairwise-stable.

Next, we show that μ is not credibly group-stable. Consider the group deviation $(A, \hat{\mu})$ from μ where $A = \{m_2, m_3, w_1, w_4\}$, $(m_2, w_4) \in \hat{\mu}$, and $(m_3, w_1) \in \hat{\mu}$. Note that both w_1 and

w_4 are matched with the best mate. Thus, the only possibility that an agent in A is strictly better off by further deviation is that either m_2 is matched with w_1 or m_3 is matched with w_4. w_1 is worse off in the former case, while w_4 is worse off in the latter case. Thus, $(A, \hat{\mu})$ is defensible. Moreover, each agent in A is better off in $\hat{\mu}$ than in μ. Hence, μ is not credibly group-stable. $\square$

Proof of Proposition 3. Fix a stable matching μ and a defensible group deviation $(A, \hat{\mu})$ from μ. Let B be the set of all agents outside A who are matched according to μ, and C be the set of all agents outside A whose partner is in A. That is, agents in B (C) satisfy condition (b) (condition (c)) in Definition 4. Note that all agents in C are unmatched at $\hat{\mu}$ and $\hat{\mu}|_B = \mu|_B$.

First, we show that if C is empty, then $\hat{\mu}$ is stable. Let C be empty. Then, $M \cup W = A \cup B$. Suppose that some agent i blocks $\hat{\mu}$. If i is in A, the blocking contradicts the defensibility of $(A, \hat{\mu})$. If i is in B, since $\hat{\mu}|_B = \mu|_B$, i blocks μ. This contradicts the stability of μ. Thus, no agent blocks $\hat{\mu}$. On the other hand, suppose that some pair (m, w) blocks $\hat{\mu}$. If either $m \in A$ or $w \in A$, then the blocking contradicts the defensibility of $(A, \hat{\mu})$. If $m \in B$ and $w \in B$, then since $\hat{\mu}|_B = \mu|_B$, (m, w) blocks μ. This contradicts the stability of μ. Hence, no pair blocks $\hat{\mu}$. Therefore, $\hat{\mu}$ is stable.

Now, to show that $\hat{\mu}$ is stable, it is sufficient to show that C is empty. Suppose for a contradiction that C is not empty. Without loss of generality, take a woman w_0 in $C \cap W$. Using the stability of μ and the defensibility of $(A, \hat{\mu})$, we will recursively construct an infinite sequence $\{(m_k, w_k)\}_{k=1}^{\infty}$ of distinct pairs in $M \times W$ such that for each $k = 1, 2, \cdots$

(a) $(m_k, w_{k-1}) \in \mu$,

(b) $(m_k, w_k) \in \hat{\mu}$,

(c) $m_k, w_k \in A$,

(d) $u_{m_k}(\mu) < u_{m_k}(\hat{\mu})$,

(e) $u_{w_k}(\mu) > u_{w_k}(\hat{\mu})$.

This contradicts the finiteness of M and W.

First, construct m_1 and w_1 that satisfy conditions (a) to (e). By the definition of group deviation, w_0 is matched with some man in A at μ. Denote this man by m_1. Thus, (a) is satisfied. Since w_0 is unmatched at $\hat{\mu}$ and μ is individually rational,

$$u_{w_0}(m_1) \equiv u_{w_0}(\mu) > u_{w_0}(\hat{\mu}) \equiv u_{w_0}(w_0), \tag{5}$$

from strict preferences. If m_1 were unmatched at $\hat{\mu}$, $u_{m_1}(w_0) \equiv u_{m_1}(\mu) > u_{m_1}(\hat{\mu}) \equiv u_{m_1}(m_1)$ by strict preferences and the individual rationality of μ. Then, the pair (m_1, w_0) would block $\hat{\mu}$, violating the defensibility of $(A, \hat{\mu})$ as m_1 is in A. Thus, it follows from the definition of group deviation that m_1 is matched with some woman in A at $\hat{\mu}$. Denote this woman by w_1. Now, $m_1, w_1 \in A$ and $(m_1, w_1) \in \hat{\mu}$ so that (b) and (c) are satisfied. Note $w_0 \neq w_1$. Since $w_0 \neq w_1$, it follows from strict preferences that

$$\text{either} \quad u_{m_1}(w_0) \equiv u_{m_1}(\mu) > u_{m_1}(w_1) \equiv u_{m_1}(\hat{\mu}), \tag{6}$$

$$\text{or} \quad u_{m_1}(w_0) \equiv u_{m_1}(\mu) < u_{m_1}(w_1) \equiv u_{m_1}(\hat{\mu}). \tag{7}$$

If the inequality (6) were true, then with the inequality (5), the pair (m_1, w_0) would block $\hat{\mu}$, violating the defensibility of $(A, \hat{\mu})$ as m_1 is in A. Thus, the inequality (7) is true so that (d) is satisfied. Now, $\mu(w_1) \neq \hat{\mu}(w_1) \equiv m_1$, otherwise we would have a contradiction that $w_0 = w_1$. Since μ is stable, it follows from the inequality (7) and strict preferences that $u_{w_1}(\mu) > u_{w_1}(\hat{\mu})$ so that (e) is satisfied. Now, $\{m_1, w_1, w_0\}$ satisfies the conditions (a) to (e).

Suppose that we are given w_0 and $\{(m_k, w_k)\}_{k=1}^{K-1}$ which satisfy conditions (a) to (e) and all of whom are distinct. We construct m_K and w_K that satisfy the conditions. First, by our hypothesis,

$$u_{w_{K-1}}(\mu) > u_{w_{K-1}}(\hat{\mu}). \tag{8}$$

If w_{K-1} were unmatched at μ, then w_{K-1} would block $\hat{\mu}$ from the inequality (8), violating the defensibility of $(A, \hat{\mu})$ as w_{K-1} is in A by our hypothesis. Thus, w_{K-1} is matched with some man at μ. Denote this man by m_K so that $(m_K, w_{K-1}) \in \mu$ and thus (a) is satisfied. Since by our hypothesis w_{K-1} is different from $w_1, \cdots, w_{K-2}$ and $(m_k, w_{k-1}) \in \mu$ for each $k = 1, \cdots, K-1$, $(m_K, w_{K-1}) \in \mu$ implies that $m_K \neq m_1, \cdots, m_{K-1}$, and thus $m_1, \cdots, m_K$ are distinct. If m_K were not in A, then m_K would be unmatched at $\hat{\mu}$ from the definition of group deviation. Then, since μ is individually rational, $u_{m_K}(w_{K-1}) \equiv u_{m_K}(\mu) > u_{m_K}(\hat{\mu}) \equiv u_{m_K}(m_K)$ from strict preferences. Thus, with the inequality (8), the pair (m_K, w_{K-1}) would block $\hat{\mu}$, violating the defensibility as w_{K-1} is in A by our hypothesis. Thus, m_K is in A. If m_K were unmatched at $\hat{\mu}$, then we would violate the defensibility like before. So, it follows from the definition of group deviation that m_K is matched with some woman in A at $\hat{\mu}$. Denote this woman by w_K so that (m_K, w_K) is in $\hat{\mu}$ and w_K is in A, and now (b) and (c) are satisfied. Since $m_1, \cdots, m_K$ are distinct and $(m_k, w_k) \in \hat{\mu}$ for each $k = 1, \cdots, K$, we have $w_K \neq w_1, \cdots, w_{K-1}$, and thus $w_1, \cdots, w_K$ are distinct. Now, because $w_{K-1} \neq w_K$, strict preferences imply that

$$\begin{aligned} &\text{either} \quad u_{m_K}(w_{K-1}) \equiv u_{m_K}(\mu) > u_{m_K}(w_K) \equiv u_{m_K}(\hat{\mu}), && (9) \\ &\text{or} \quad u_{m_K}(w_{K-1}) \equiv u_{m_K}(\mu) < u_{m_K}(w_K) \equiv u_{m_K}(\hat{\mu}). && (10) \end{aligned}$$

If the inequality (9) were true, then with the inequality (8), the pair (m_K, w_{K-1}) would block $\hat{\mu}$, violating the defensibility as m_K and w_{K-1} are in A. Thus, the inequality (10) holds so that (d) is satisfied. Finally, $\mu(w_K) \neq m_K$ as m_K is matched with $w_{K-1} \neq w_K$ at μ. This implies from the stability of μ and the inequality (10) that $u_{w_K}(\mu) > u_{w_K}(\hat{\mu}) \equiv u_{w_K}(m_K)$ so that (e) is satisfied. Now, we have the desired sequence. □

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